Reprinted from SOIL SCIENCE SOCIETY OF AMERICA PROCEEDINGS
Vol. 27, No. 3, May-June 1963, pages 263-265
677 South Segoe Road, Madison 11, Wisconsin

Influence of Variations in the Diffusivity-Water Content Relation on Infiltration[1]

R. J. Hanks and S. A. Bowers[2]

ABSTRACT

Estimates of the influence of variations in the diffusivity (or conductivity)-water content relation on infiltration were made using a numerical method. Where the diffusivity was increased or decreased up to a factor of two at saturation, infiltration was markedly changed. However, variations of the diffusivity by a factor of 100 at drier moisture contents had no significant influence on infiltration. This implies that infiltration is governed, to a large extent, by the soil properties at water contents near saturation and is little influenced by soil properties at drier water contents.

The flow equation,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial H}{\partial x}\right)$$

where θ is volumetric water content, H is hydraulic head, x is distance, t is time, and K is capillary conductivity, has been employed by many workers (2, 5, 7) to describe the flow of water in soils. While the equation is limited to problems where hysteresis and shrinkage or swelling are not important, it has given good approximations of many infiltration problems.

The flow equation is difficult to solve because the equation is nonlinear. Recent advances have been made in its solution, however. Scott et al. (7) described a power series solution for horizontal infiltration where the diffusivity-water content relation is linear or exponential. Philip (6) described a numerical solution for vertical infiltration for a homogeneous soil with a uniform initial moisture content. Hanks and Bowers (4) described a numerical method, which places no restrictions on the initial water content or diffusivity-water content relation, to solve vertical infiltration into layered soils. Thus, methods are available to solve the flow equation for almost any infiltration problem where the equation applies.

Solution of the flow equation for infiltration requires knowledge of the pressure head(tension)-water content relation as well as knowledge of the diffusivity(or conductivity)-water content relation. The measurement of these two relations is extremely difficult and is influenced by many factors difficult to control. In addition, the conductivity for most soils is commonly 1,000 to 10,000 times greater at saturation than at water contents found at the initiation of infiltration. Thus, it was hypothesized that careful measurements of conductivity and pressure head at water contents near saturation would be needed to estimate infiltration but crude measurements may be sufficient at water contents commonly found at the start of infiltration. This experiment was conducted to test this hypothesis.

PROCEDURE

The numerical method described by Hanks and Bowers (4) and IBM 650 and 1620 high-speed digital computers were used in this study. Computations were made for infiltration into a soil where all treatments had identical boundary and initial conditions. The only differences among treatments were in the diffusivity-water content relation.

Gardner (2) indicated that for many soils the diffusivity can be related to the water content by an exponential relation of the form $D = ae^{\beta\theta}$ where D is the diffusivity, θ is the water content, a is the diffusivity at $\theta = 0$, and β is a constant. This exponential relation was used as the "standard" about which the diffusivity-water relation was varied. The data actually used were for Sarpy loam (4), adjusted slightly to conform to D (cm.² per sec.) $= 1.24 \times 10^{-4} e^{19.78\theta}$.

The diffusivity can be varied by varying either the conductivity, K, or the specific water capacity, $d\theta/dh$, in accordance with the relation $D = K\, dh/d\theta$, where h is pressure head. For most of the treatments the specific water capacity was held constant. This is equivalent to maintaining a constant pressure head-water content relation. Under these conditions, variations in the diffusivity-water content relation are equivalent to variations in the conductivity-water content relation. This was done because conductivity was believed to be the more difficult factor to measure and the one more subject to variation. However, for illustration two treatments involved changes in the specific water capacity with the conductivity held constant.

The treatments where the diffusivity was varied by varying conductivity are shown in figure 1 and are as follows:

A. Diffusivity related to water content by the relation, $D = 1.24 \times 10^{-4} e^{19.78\theta}$. ABCD of figure 1.
B. Same as A except diffusivity increased up to a factor of 10 at $\theta = 0.061$. EBCD of figure 1.
C. Same as A except diffusivity increased up to a factor of 100 at $\theta = 0.061$. GKCD of figure 1.
D. Same as A except diffusivity decreased up to a factor of 10 at $\theta = 0.061$. FBCD of figure 1.
E. Same as A except diffusivity increased up to a factor of 2 at $\theta = 0.41$. ABCH of figure 1.

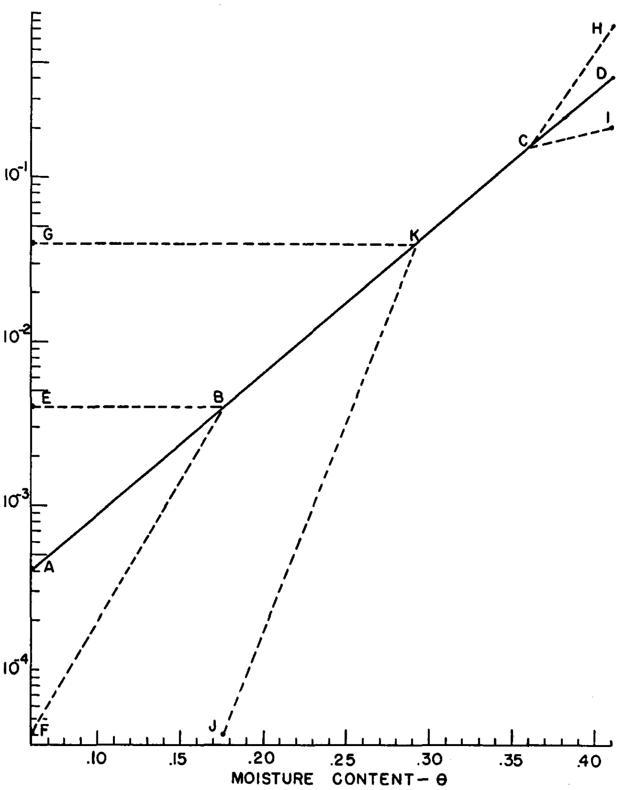

Figure 1—Diffusivity-water content relations studied.

[1]Contribution from the Soil and Water Conservation Research Division, ARS, USDA, in cooperation with the Kansas Agr. Exp. Sta. Contribution No. 789, Department of Agronomy, Kansas Agr. Exp. Sta., Manhattan. Supported in part by North Central States Regional Committee NC-40. Received June 15, 1962. Approved Jan. 21, 1963.

[2]Soil Scientists (Physics), USDA, Manhattan, Kans.

F. Same as A except diffusivity decreased up to a factor of 2 at $\theta = 0.41$. ABCI of figure 1.

Computations were made on all treatments for vertical infiltration into a semi-infinite column of soil for the following initial and boundary conditions:

$$\theta = 0.41 \text{ for } x = 0,\ t > 0$$
$$\theta = 0.61 \text{ for } x > 0,\ t = 0.$$

Computations also were made for the ABCH, ABCD, ABCI, and GKCD treatments for the following conditions:

$$\theta = 0.410 \text{ for } x = 0,\ t > 0$$
$$\theta = 0.175 \text{ for } x > 0,\ t = 0.$$

Computations also were made for treatment JKCD of figure 1. Diffusivity was 1/100 that of ABCD at $\theta = 0.175$. The pressure head-water relation used for all of the above comparisons was curve XWZ of figure 2.

The following treatments involved a change in the pressure head-water content relation with the conductivity-water content relation constant:

A. Curve XWZ of figure 2. Basic curve.

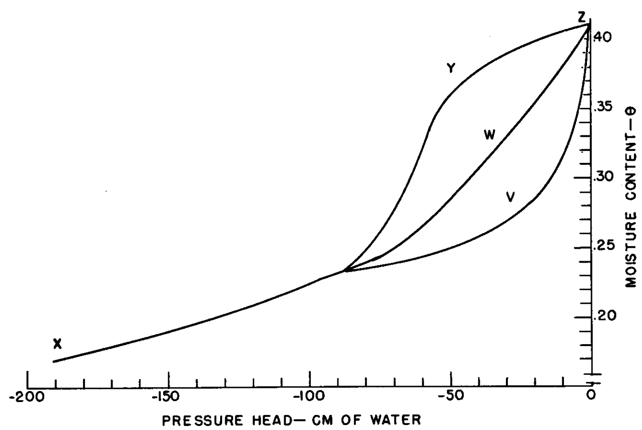

Figure 2—Pressure head-water content relations studied.

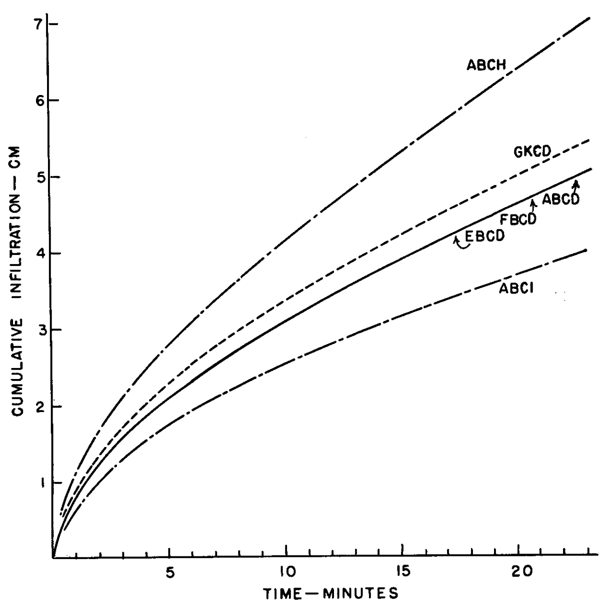

Figure 3—Cumulative infiltration-time curves for several diffusivity-water content relations. The boundary and initial conditions were $\theta = 0.41$ at $x = 0$, $t > 0$, $\theta = 0.061$ at $t = 0$, $x > 0$. The same pressure head-water content data applied to all curves.

B. Curve XYZ of figure 2. Low specific moisture capacity (high diffusivity) on wet end.

C. Curve XVZ of figure 2. High specific moisture capacity (low diffusivity) on wet end.

Computations were made for these conditions for the second set of boundary and initial conditions given above.

RESULTS AND DISCUSSION

Preliminary computations showed that there was no measurable difference in infiltration using the actual data reported in (4) compared to exponential data for Sarpy loam provided the initial and boundary conditions were the same. Preliminary tests were also conducted using the data in (4) for Geary silt loam. The results were almost identical, in general, to those for Sarpy loam. Examination of the diffusivity-water content data for Geary silt loam showed that an exponential relation gave a good

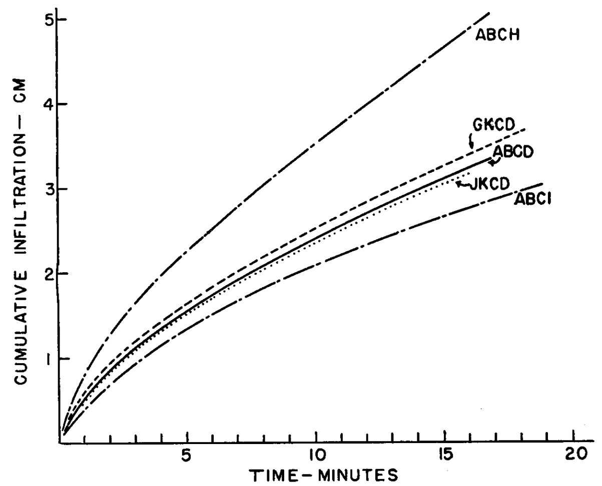

Figure 4—Cumulative infiltration-time curves for several diffusivity-water content relations. The boundary and initial conditions were $\theta = 0.41$ at $x = 0$, $t > 0$, $\theta = 0.175$ for $t = 0$, $x > 0$. The same pressure head-water content data applied to all curves.

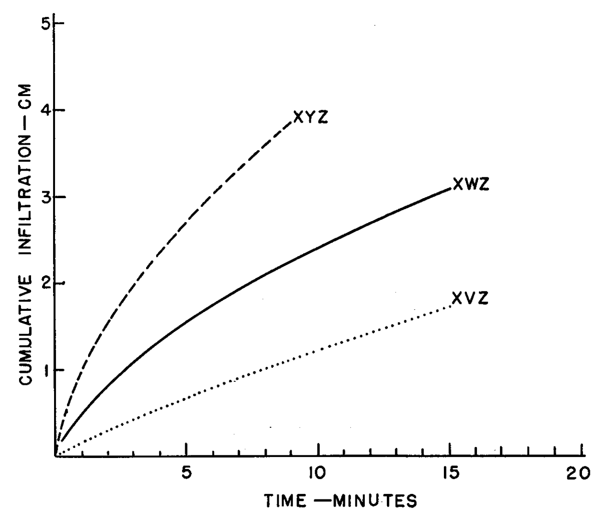

Figure 5—Cumulative infiltration-time curves for the pressure head-water content relations of figure 2. The initial and boundary conditions were $\theta = 0.41$ at $x = 0$, $t > 0$, $\theta = 0.175$ for $t = 0$, $x > 0$. The same conductivity-water relations data applied to all curves.

fit with the constant β equal to 21.0. This value of β for Geary silt loam was only slightly higher than the value of $\beta = 19.78$ found for Sarpy loam. If the values of β are the same for two different soils, computations made similar to those reported here will give similar results. The only difference will be that the plots of cumulative infiltration vs. time will be scaled by a constant factor related to the value of "a." Thus, it is reasonable that computations for Sarpy loam and Geary silt loam gave similar results because their respective values of β were about the same. It is interesting to note that the values of β reported by Gardner and Hillel (3) (their α) were 18.5 for Pachappa sandy loam and 18.2 to 20.8 for Indio loam. Therefore, it was concluded that detailed computations would be made only on the relations shown in figures 1 and 2. The results should be applicable to many other soils where the values of β and initial conditions are similar.

The influence of the variations in the diffusivity-water content relation on cumulative infiltration as a function of time is shown in figures 3, 4, and 5. Figures 3 and 4 show the influence of changes in the conductivity relation on cumulative infiltration, and figure 5 shows the results of changing the specific water capacity relation.

When the conductivity was varied on the wet end, the data of figures 3 and 4 show that the cumulative infiltration was influenced markedly. After 15 minutes of infiltration, for the conditions of figure 3, the cumulative infiltration for curve ABCH was about 1.37 times that of ABCD and the cumulative infiltration for curve ABCI was about 0.80 times that of ABCD. After 15 minutes of infiltration, for the conditions of figure 4, the cumulative infiltration for curve ABCH was about 1.46 times that of curve ABCD and the cumulative infiltration for curve ABCI was about 0.85 times that of ABCD.

When conductivity was varied on the dry end, much less influence on cumulative infiltration resulted. Figure 3 shows that there were no appreciable differences among treatments FBCD, EBCD, and ABCD despite the diffusivity being different by a factor of 100 on the dry end. Treatment GKCD gave slightly higher infiltration for both boundary conditions.

For the second boundary condition where the initial water content was 0.175, the results of figure 4 show that great variation in the diffusivity (conductivity) on the "dry" water content region had little influence on cumulative infiltration. Treatment JKCD, which had a diffusivity 1/100 that of ABCD at $\theta = 0.175$, gave only slightly lower cumulative infiltration. Treatment GKCD resulted in slightly higher infiltration than ABCD. In general, the results for the two different initial conditions lead to the same conclusion—that changes in the diffusivity-water relation near saturation have a large influence on infiltration but progressively less influence as moisture contents become drier.

The data of figures 3 and 4 agree with the analysis of Crank (1) for horizontal infiltration. Crank reported that the cumulative infiltration, Q, was given by

$$Q = 2\,(\theta_s - \theta_o)\,\sqrt{\bar{D}t/\pi}$$

where θ_s is the water content at the source, θ_o is the initial water content and $\bar{D}$ is a weighted mean diffusivity. The weighted mean diffusivity is given by

$$\bar{D} = \frac{5}{3(\theta_s - \theta_o)^{5/3}} \int_{\theta_o}^{\theta_s} (\theta - \theta_o)^{2/3}\, D(\theta)d\theta$$

The weighted mean diffusivity gives greater weight to the values of $D(\theta)$ near θ_s (wet end). Table 1 shows the value of $\bar{D}$ calculated by the method of Crank together with the cumulative infiltration after 15 minutes for the diffusivity-water content relations shown in figures 3 and 4. The horizontal infiltration computed by use of the above equations is also shown.

The data of table 1 show that the variations in the diffusivity on the dry end had only a slight influence on the weighted mean diffusivity. The weighted mean diffusivity was influenced markedly by variations near θ_s. This is because the weighting term $(\theta - \theta_o)^{2/3}$ is greater near θ_s than near θ_o as well as because $D(\theta)$ is much larger near θ_s. Thus, the conclusions based on the mean weighted diffusivity analysis of Crank (1) for *horizontal infiltration* agrees with the analysis reported herein for *vertical infiltration.*

A comparison (table 1) of the cumulative infiltration computed for horizontal infiltration with that for vertical infiltration shows, with one exception, that infiltration was less for horizontal than for the vertical computation. This is the expected result. The vertical computation indicates a greater relative difference between extreme "treatments" for both values of θ_o.

The data of figure 5 show that infiltration can be greatly influenced by a change in the pressure-water content relation with no change in the conductivity-water content relation. For treatment XYZ, where the specific moisture capacity was low on the wet end (diffusivity high), the infiltration was higher than it was with base data treatment XWZ. Conversely, treatment XVZ, where the specific moisture capacity was highest on the wet end (diffusivity low), decreased infiltration.

Thus the data indicate that an accurate knowledge of soil moisture diffusivity near saturation is necessary for accurate predictions of infiltration. However, an accurate knowledge of diffusivity for the drier water contents does not appear to be necessary. While these conclusions apply strictly only to soils having characteristics and for boundary conditions similar to the one tested, the analysis of Crank (1) indicates the conclusions are general.

Table 1—Comparison of the cumulative infiltration calculated by the method of Crank (1) with the numerical method.

Diffusivity-water content relation	$\bar{D}$ (Crank)	Cumulative infiltration after 15 minutes: Horizontal (Crank)	Vertical
		$\theta_o = 0.175$	
ABCH	0.2084 cm.²/sec.	3.64 cm.	4.62 cm.
GKCD	0.1480	3.07	3.23
ABCD	0.1417	3.00	3.08
JKCD	0.1381	2.96	3.03
ABCI	0.1053	2.59	2.63
		$\theta_o = 0.061$	
ABCH	0.1397 cm.²/sec.	4.43 cm.	5.35 cm.
GKCD	0.1099	3.94	4.27
EBCD	0.0967	3.69	3.98
ABCD	0.0964	3.68	3.97
FBCD	0.0963	3.68	3.97
ABCI	0.0727	3.20	3.17

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